Session 5: Torque and Statics

Part A- Torque and rotational motion
Part B- Equilibrium Part C: Free body diagrams

Dr. Arshiya Sood

June 07, 2025

Table of Contents

Session 4 Recap & Reflect: What Did We Learn?

Motion

- **Motion is fundamental** to robotic behavior — it enables sensing, interacting, and acting.
- We studied two types of motion:
  - **Linear motion**: Movement along a straight line.
  - **Angular motion**: Rotation about a fixed axis.
- Most robots involve both types — e.g., a wheeled robot moves forward (linear) while its wheels rotate (angular).
- We covered:
  - Position, displacement, velocity, acceleration (linear and angular)
  - Rigid body motion
  - Moment of inertia
  - Tangential and radial motion components
  - Linking angular motion to linear effects

Linear vs Angular Motion

Quantity	Linear Motion	Angular Motion
Position	x: position along a line	θ: angular position
Displacement	$\Delta x = x_f - x_i$	$\Delta\theta = \theta_f - \theta_i$
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
Inertia	Mass m	Moment of inertia I
Force Equivalent	Force $F = ma$	— (torque not yet introduced)

Linear and angular motions mirror each other — this parallel helps transfer intuition in robotics.

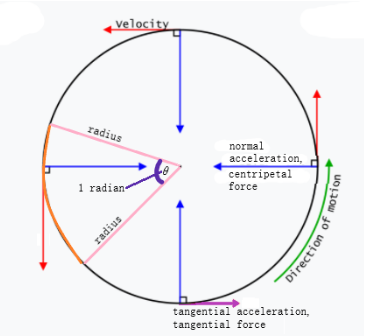
Linear vs Tangential Velocity

Aspect	**Linear Velocity**	**Tangential Velocity**
Meaning	Rate of change of position along a line	Linear speed of a point on a rotating object
Applies to	Translating (non-rotating) bodies	Rotating bodies (rigid)
Formula	$v = \frac{dx}{dt}$	$v = r\omega$
Direction	Along the path of motion	Tangent to the circular path
Variation across body	Same for all points (rigid translation)	Varies with radius r from axis
Units	m/s	m/s

Tangential velocity is a specific case of linear velocity for points on a rotating object.

Acceleration in Circular Motion

- **Tangential Acceleration** $a_t = r\alpha$
  - Due to change in angular speed ($\alpha \neq 0$).
  - Direction: tangent to circular path.
- **Centripetal (Radial) Acceleration** $a_r = \omega^2 r$
  - Due to change in direction during rotation.
  - Direction: toward center of rotation.



Total Acceleration:

$$a = \sqrt{a_t^2 + a_r^2}$$

Both components act simultaneously in circular motion unless $\omega = 0$.

Tangential vs Centripetal Acceleration

Aspect	Tangential Acceleration a_t	Centripetal Acceleration a_r
Meaning	Change in speed of a rotating point	Change in direction of velocity during circular motion
Formula	$a_t = r\alpha$	$a_r = \frac{v^2}{r} = \omega^2 r$
Direction	Tangent to the circular path	Always toward the center of rotation
Exists when	Angular velocity ω is changing (i.e., $\alpha \neq 0$)	Any time an object is moving in a circle, even at constant speed
Zero when	$\alpha = 0$ (constant speed rotation)	$\omega = 0$ (no rotation)

Both accelerations are experienced simultaneously in circular motion when speed and direction are changing.

Session 4 : Recap & Reflect (contd.)

What is Moment of Inertia?

- It measures an object's resistance to angular acceleration.
- It depends on:
  - The object's mass.
  - How far the mass is distributed from the axis of rotation.
- Unit: $\mathrm{kg} \cdot \mathrm{m}^2$

Formula:

- For a point mass: $I = mr^2$
- For multiple point masses: $I = \sum m_i r_i^2$
- For continuous bodies: $I = \int r^2 \, dm$

Greater the distance of mass from the axis, larger the moment of inertia.

Session 4 Recap Quiz

Recap Quiz

Q1. A point on the rim of a rotating wheel moves with constant angular velocity ω. Which of the following is true about its acceleration?

1. $a = 0, \ \alpha = 0$
2. $a \neq 0, \ \alpha = 0$
3. $a = 0, \ \alpha \neq 0$
4. $a \neq 0, \ \alpha \neq 0$

Q2. What is the direction of the net acceleration of a point on a rotating disc when both angular speed and angular acceleration are non-zero?

1. Radial (inward)
2. Tangential
3. Perpendicular to the plane of rotation
4. At an angle between radial and tangential directions

Recap Quiz (contd.)

Q3. A robot arm lifts a payload and stretches forward to place it on a shelf. The mass is small, but the shelf is far. Why is the moment of inertia highest when the arm is fully extended? Because

1. mass increases as the arm extends
2. angular velocity is higher at full reach
3. mass is farther from the rotation axis
4. tangential velocity is zero at the tip

Q4. A robot's sensor rotates back and forth for scanning. It starts from rest, accelerates to a fixed angular speed, and then decelerates to stop after each scan cycle. When is tangential acceleration non-zero?

1. Only during the mid-scan when speed is constant
2. Only during start and stop
3. Throughout the motion
4. Never

Recap Quiz (contd.)

Q5. The tip of a robot arm is moving along a circular arc at a constant angular velocity. At some point, the gripper releases an object while rotating. What kind of path will the object follow immediately after release? It will

1. fall straight down
2. continue along the circular path
3. move tangentially to the arc at the point of release
4. spiral outward from the arm

Q6. A self-driving car is moving at a constant speed while turning around a wide curve. The onboard sensors detect acceleration.What kind of acceleration is the car experiencing?

1. Tangential acceleration
2. No acceleration
3. Centripetal acceleration
4. Linear forward acceleration

Recap Quiz (contd.)

Q7. A humanoid swings its arms while walking. The arms swing like pendulums from the shoulder joint. What happens to tangential velocity of the hand?

1. It stays constant throughout the motion
2. It increases when angular velocity increases
3. It decreases with distance from the shoulder
4. It is unrelated to angular velocity

Q8. A mobile robot carries a 5 kg payload. Initially, the payload is mounted at the center and later moved to the back edge of the robot. What happens to its moment of inertia when turning?

1. It increases
2. It decreases
3. It stays the same
4. The robot moves faster

Answer Key with Explanations

- **Q1: B** — Centripetal acceleration exists due to direction change, but ω is constant so $\alpha = 0$.
- **Q2: D** — Both tangential (a_t) and radial (a_r) acceleration are present. The net acceleration is the vector sum — angled between them.
- **Q3: C** — Moment of inertia $I = \sum mr^2$ increases with distance. Fully extended arm increases r, hence higher inertia.
- **Q4: B** — Tangential acceleration exists only when angular velocity changes — i.e., at start and stop phases.
- **Q5: C** — Once released, the object moves in the direction of its tangential velocity — traight line tangent to the circle at the release point.
- **Q6: C** — Even if speed is constant, turning changes direction, causing inward (centripetal) acceleration.
- **Q7: B** — As the robot's arm swings faster, angular speed increases. Since, $v = r\omega$, the tangential speed of the hand increases too.
- **Q8: A** — Moment of inertia increases when mass is farther from the axis. Shifting the payload from center to edge increases radius r, thus increasing inertia $I = mr^2$.

Part A- Torque

Torque and Rotational Motion: The Rotational Equivalent of Force

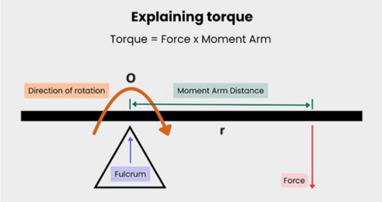
What is Torque?

Just as force changes an object's translational motion, **torque** changes an object's *rotational* motion about a pivot or axis.

Torque is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \Rightarrow \quad \tau = rF \sin\theta$$

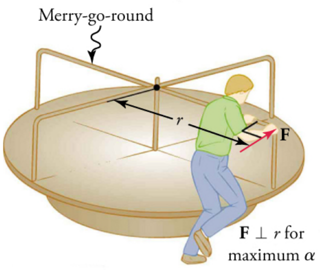
- $\vec{\tau}$ is the torque vector (in Nm),
- $\vec{r}$ is the lever arm — the position vector from the pivot point (O) to the point where the force is applied.,
- $\vec{F}$ is the applied force,
- θ is the angle between $\vec{r}$ and $\vec{F}$.



Direction of Torque: Use the *Right-Hand Rule* — curl your fingers from $\vec{r}$ to $\vec{F}$; your thumb gives the direction of $\vec{\tau}$.
Torque is a vector and points along the axis of rotation.

Everyday Example – Merry-Go-Round

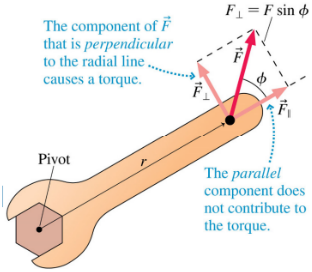
- Torque increases when force is applied farther from the pivot.
- Larger lever arm = more torque = faster rotation.



Another Way of Expressing Torque

$$\tau = r \cdot F_{\perp} = r \cdot F_{\tan}$$

- $F_{\perp}$ or $F_{\tan}$: Component of force perpendicular to the lever arm.
- Tangential force $F_{\tan}$ causes rotation.
- Radial force F_{rad} passes through pivot → no torque.
- Angle ϕ is between $\vec{r}$ and $\vec{F}$.



Types of Torque: Static vs Dynamic

Static Torque

A twisting force applied to an object that does **not cause it to rotate**. It arises when the object is constrained or held in place. Even though no motion occurs, muscles or mechanisms must still exert force to maintain the torque.

Examples:

- Pushing a closed door that doesn't move.
- Holding a wrench still while tightening a bolt.

Dynamic Torque

A twisting force that **results in rotation** and produces angular acceleration. It causes motion and is commonly associated with engines, motors, or rotating mechanical systems.

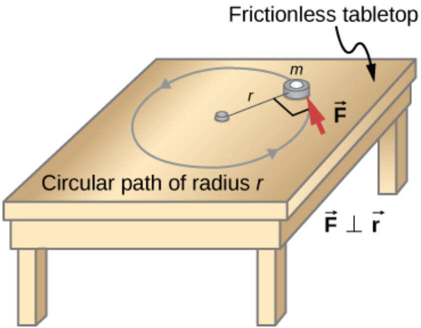
Examples:

- A car engine turning the wheels.
- A motor shaft spinning.

Rotational Dynamics – Newton's Second Law

What is the rotational counterpart of Newton's Second Law?

- Start with the translational form: $\sum \vec{F} = m\vec{a}$
- Consider a point mass m rotating at radius r, with a tangential force $\vec{F}$
- From linear dynamics: $F = ma$
- For circular motion: $a = r\alpha$
- Substituting: $F = mr\alpha$
- Multiply both sides by r: $rF = mr^2\alpha$
- Recognize:
  - Torque: $\tau = rF$
  - Moment of inertia: $I = mr^2$
- Therefore: $\tau = I\alpha$

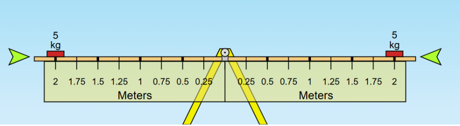


For a rigid body: $\sum \tau_i = I\alpha$

Torque Analysis – Balanced Cases

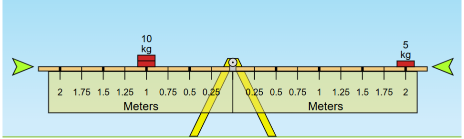
Example 1: Equal Masses, Equal Distances

- 5 kg at +2 m (right), 5 kg at −2 m (left)
- Left: $\tau = -2 \cdot (5 \cdot 9.8) = -98\,\text{Nm}$
- Right: $\tau = +2 \cdot (5 \cdot 9.8) = +98\,\text{Nm}$
- Net Torque: $0 \rightarrow$ **Perfectly Balanced**



Example 2: Unequal Masses, Adjusted Distance

- 10 kg at +1 m, 5 kg at −2 m
- Left: $\tau = -2 \cdot (5 \cdot 9.8) = -98\,\text{Nm}$
- Right: $\tau = +1 \cdot (10 \cdot 9.8) = +98\,\text{Nm}$
- Net Torque: $0 \rightarrow$ **Balanced**



Simulation source:
`https://phet.colorado.edu/sims/html/balancing-act/latest/balancing-act_en.html`PhET Balancing Act Simulation

Torque Analysis – Unbalanced Case

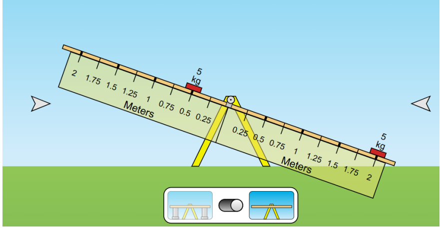
Example 3: Same Masses, Unequal Distances

- Place a 5 kg mass at $+2\,\text{m}$ (right side)
- Place another 5 kg mass at $-0.5\,\text{m}$ (left side)

Torque Calculation:

- Left: $\tau = -0.5 \cdot (5 \cdot 9.8) = -24.5\,\text{Nm}$
- Right: $\tau = +2 \cdot (5 \cdot 9.8) = +98\,\text{Nm}$
- Net Torque: $+73.5\,\text{Nm}$

Conclusion: The beam rotates **clockwise** due to stronger torque on the right.



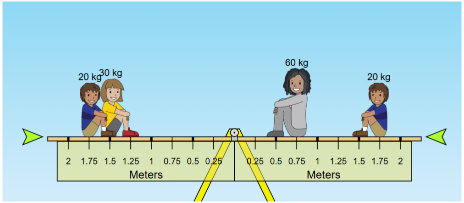
Multi-Object See-Saw Torque Analysis

Left Side (Negative Positions):

- 30 kg at $-1.5\,\text{m}$:
  $\tau = -1.5 \cdot (30 \cdot 9.8) = -441\,\text{Nm}$
- 20 kg at $-1.75\,\text{m}$:
  $\tau = -1.75 \cdot (20 \cdot 9.8) = -343\,\text{Nm}$

Right Side (Positive Positions):

- 60 kg at $+0.75\,\text{m}$:
  $\tau = +0.75 \cdot (60 \cdot 9.8) = +441\,\text{Nm}$
- 20 kg at $+1.75\,\text{m}$:
  $\tau = +1.75 \cdot (20 \cdot 9.8) = +343\,\text{Nm}$



Net Torque:

$$-441 - 343 + 441 + 343 = \mathbf{0}\,\text{Nm}$$

Conclusion: The system is in *rotational equilibrium*.

Torque Calculation – Revolute Joint in a Robot Arm

Suppose you have a robot with a **revolute joint** and an arm of **0.5 meters**. The robot is lifting a **2 kg load** at a **joint angle of 30°**. What is the torque required for the joint?

Step 1: Compute Load Weight

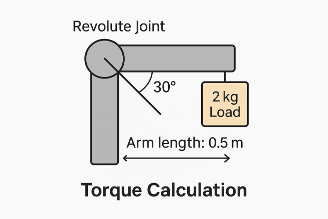
$$F = m \cdot g = 2 \cdot 9.81 = 19.62\,\mathrm{N}$$

Step 2: Use Torque Formula

$$\tau = r \cdot F \cdot \sin(\theta)$$

$$\tau = 0.5 \cdot 19.62 \cdot \sin(30^\circ) = 0.5 \cdot 19.62 \cdot 0.5$$

$$\tau = \mathbf{4.905\ Nm}$$



A robot arm lifting a load at angle $\theta = 30^\circ$

The required torque is **4.905 Newton-meters**.

Torque in Action – Simulations & Real-World Examples

PhET Simulation: Balancing Act

- Try it yourself: https://phet.colorado.edu/en/simulation/balancing-act
- Observe torque on each side and net torque in real-time.
- Practice balancing by adjusting weight and arm length.
- Excellent for reinforcing concepts like lever arms and rotational equilibrium.

Everyday Examples of Torque

- Watch here: https://www.youtube.com/watch?v=-yXPReR-31E
- Demonstrates torque and types of torque using familiar objects: wrenches, see-saws, bicycles.
- Builds real-world intuition around the torque formula.

Torque in Boston Dynamics' Spot Robot

- Watch here: https://www.youtube.com/watch?v=tfWbE$_1eCZk$
- Shows how Spot uses joint torques for walking and balancing.
- Highlights precise motor control and torque feedback in real-time robotic motion.

Part B- Equilibrium

Why Equilibrium is Important in Robotics

- **Foundation for Stability and Safety:**
  Robots must maintain balance while handling loads or operating on uneven terrain.
  Example: A humanoid robot lifting a box must adjust posture to avoid falling.
- **Enables Accurate Control:**
  Control algorithms (e.g., PID) rely on initial balance to execute precise movements.
  Example: A manipulator arm holding a tool steady before welding.
- **Necessary for Grasping and Holding Objects:**
  Grippers must counteract forces and torques to prevent slipping.
- **Crucial for Walking and Dynamic Motion:**
  Walking robots use real-time force/torque balance to stay upright.
  Example: Spot robot from Boston Dynamics adjusting to terrain.
- **Avoiding Failures:**
  Poor equilibrium causes overloads, instability, and mechanical failure.

Understanding Mechanical Equilibrium

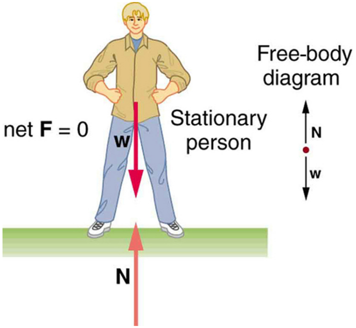
What is Equilibrium?

- A state of balance where opposing forces or influences cancel out.
- Can be at rest or in steady motion — no net force or torque.
- Includes both static and dynamic conditions.

Types of Mechanical Equilibrium:

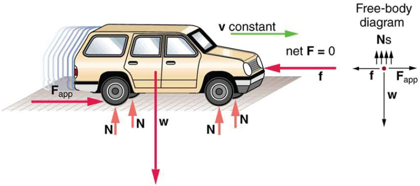
Static Equilibrium

- Object is at rest: $\sum \vec{F} = 0$, $\sum \tau = 0$
- No linear or angular acceleration
- *Examples:*
  - Book on a table
  - Person standing still
  - Balanced seesaw



Dynamic Equilibrium

- Object is in motion with constant velocity: $\sum \vec{F} = 0$, $\sum \tau = 0$
- No net acceleration or change in angular velocity
- *Examples:*
  - Car at constant velocity
  - Skydiver at terminal speed
  - Pendulum at constant amplitude

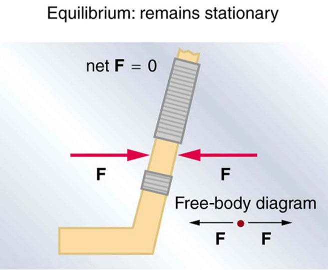


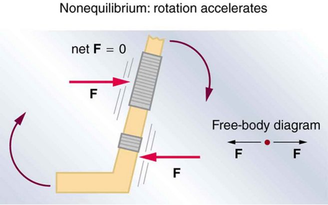
A

system with zero net force is **not necessarily in static equilibrium**

Condition 2: Net Torque Must Be Zero

- **Example: Hockey Stick on Ice**
  - In **Figure A**, two equal and opposite forces are applied at the same point on the stick.
  - $\Rightarrow$ Net force $= 0$ & Net torque $= 0 \Rightarrow$ No motion — Static equilibrium achieved.
  - In **Figure B**, same forces applied at **different points**.
  - $\Rightarrow$ Net force $= 0$ but Net torque $\neq 0 \Rightarrow$ Stick rotates — Not in equilibrium.
- **Conclusion:**
  - For complete equilibrium, both conditions must be satisfied:
    - $\sum \vec{F} = 0$ (No net force)
    - $\sum \tau = 0$ (No net torque)
  - Torque depends on *magnitude, direction, and point of application*.



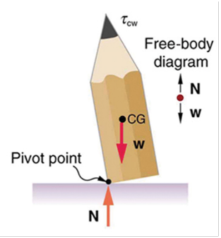


Stability of Equilibrium – Stable vs Unstable

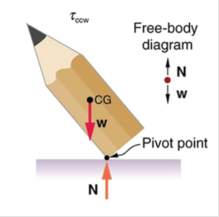
Stability: Describes how the system behaves when slightly disturbed.

Types of Stability

- **Stable Equilibrium:**
  - Restoring torque brings system back to equilibrium.
  - *Example:* A slight counterclockwise tilt causes a restoring clockwise torque due to the pencil's weight, bringing it back to equilibrium.

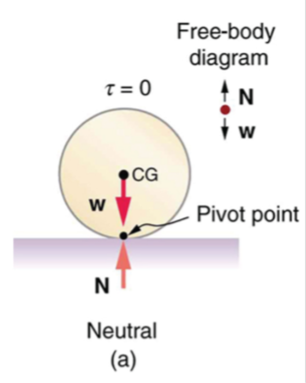


- **Unstable Equilibrium:**
  - A system in unstable equilibrium accelerates away from its equilibrium position.
  - *Example:* If the pencil is displaced too far, the torque caused by its weight changes to counterclockwise and causes the displacement to increase.



Neutral Equilibrium

- System stays in new position after displacement.
- No net force or torque to return or move it further.
- Equilibrium is independent of position.
- *Example:* Ball resting on a flat horizontal surface.



Source:
https://uta.pressbooks.pub/oert-mpsfundamentals/chapter/chapter-6-basic-dynamics-and-static-equilibrium/
https://www.openassembly.com/document/6d370d32-99fc-4dc0-b06c-210f400afa31?context=

What is a Free-Body Diagram (FBD)?

- **Definition:**
  - A **Free-Body Diagram (FBD)** is a simplified sketch that shows all the **external forces and torques** acting on a single object.
  - It helps isolate the object from its environment, focusing only on what influences its motion or balance.
  - Think of it as: *"Zooming in on one object to analyze what's pushing or pulling on it."*

- **Why Use FBDs?**
  - Clarifies complex mechanical situations by simplifying them.
  - Helps apply Newton's Laws ($\sum \vec{F} = 0$, $\sum \tau = 0$, or $\sum \vec{F} = m\vec{a}$).
  - Essential for solving equilibrium and dynamics problems.
  - Helps identify incorrect assumptions or missing forces.
  - Bridges the gap between physical intuition and mathematical formulation.

FBDs in Robotics

- **Why FBDs Matter in Robotics:**
  - FBDs help calculate the **torque** required at joints to lift, hold, or move objects.
  - They aid in identifying **external forces** such as gravity, payloads, or ground reaction forces.
- **Without FBDs:**
  - Force and torque calculations become guesswork.
  - Increased risk of incorrect actuator sizing or overload.

FBDs in Robotics

- **Applications of FBDs:**
  - Crucial for analyzing **robot stability** (e.g., humanoid balance or quadrupeds).
  - Assists in determining structural **load paths** and stress points.
  - Supports **energy-efficient design** by minimizing unnecessary force exertion.
- **Consequences of Skipping FBDs:**
  - Potential **mechanical failure** or unstable gait/movement.
  - Lower accuracy in motion planning and force control.